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Extended Cesàro operators on Bergman spaces ☆

Zhangjian Hu

Department of Mathematics, Huzhou Teachers College, Huzhou, Zhejiang 313000, PR China
Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, USA

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Abstract

We define an extended Cesàro operator T_g with holomorphic symbol g in the unit ball B of C^n. For a large class of weights w we characterize those g for which T_g is bounded (or compact) from Bergman space $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$, $0 < p, q < \infty$. In addition, we obtain some results about equivalent norms, the norm of point evaluation functionals, and the interpolation sequences on $L^p_{a,w}(B)$.

Keywords: Cesàro operator; Bergman space; Normal weight

1. Introduction

Let D be the unit disk in the complex plane C. For a holomorphic function $f(z)$ on D with Taylor expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the Cesàro operator acting on f is

$$C[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.$$

With the result of Hardy [7] and M. Riesz's theorem [5] we know that $C[\cdot]$ is bounded on $H^p(D)$ for $1 < p < \infty$. In [16] and [9] it is proved that $C[\cdot]$ is bounded on $H^p(D)$ for $p = 1$ and $0 < p < 1$, respectively, where $H^p(D)$ is the Hardy space on D. A little calculation shows that $C[f](z) = \frac{1}{z}\int_0^z f(t)(\log(1/(1-t)))'\,dt$. Hence, on most holo-

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E-mail addresses: huzj@mail.huptt.zj.cn, huz@email.unc.edu.

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morphic function spaces, $C[f]$ is bounded if and only if the integral operator $f \mapsto \int_0^z f(t)(\log(1/(1-t)))'\,dt$ is bounded. From this point of view it is natural to consider the extended Cesàro operator T_g, depending on an holomorphic symbol g, as follows:

$$T_g f(z) = \int_0^z f(t)g'(t)\,dt. \tag{1}$$

In [10] Pommerenke proved T_g is bounded on $H^2(D)$ if and only if g is a BMOA function. Aleman and Siskakis generalized this result to $1 \leqslant p < \infty$ in [2]. Recently, Aleman and Cima obtained the solution for the problem: given $p, q \in (0, \infty)$ characterize the symbols g for which T_g maps $H^p(D) \to H^q(D)$ boundedly. See [1]. In the Bergman space setting, for $p \geqslant 1$ Xiao proved $C[f]$ is bounded on $A^p(\varphi^p(r)/(1-r))$ with $\varphi > 0$ as in [18]. Aleman and Siskakis studied T_g on $L_a^p(w)$ with a large class of other weights w. Restricting themselves to $p \geqslant 1$ they obtained a necessary and sufficient condition on g for which T_g is bounded on $L_a^p(w)$ in [3].

The purpose of this paper is to define the extended Cesàro operator on the unit ball B of C^n and to characterize those holomorphic symbols for which the induced operator is bounded (or compact) from pth Bergman space to qth Bergman space.

For later use we need some more notation. We denote by $H(B)$ the class of all holomorphic functions on the unit ball B of C^n. For $g \in H(B)$ let $\Re g(z) = \sum_{j=1}^n z_j(\partial g/\partial z_j)$ be the radial derivative.

Definition. Given $g \in H(B)$, the extended Cesàro operator T_g with symbol g is the operator on $H(B)$,

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\,\frac{dt}{t}, \quad f \in H(B),\ z \in B. \tag{2}$$

It is clear that when $n = 1$, (2) is just (1).

As in [3], a positive Lebesgue integral function w on the interval $[0, 1)$ is called normal if there are two positive constants C_1, C_2 and some $s \in (0, 1)$ such that for all $r \in (0, 1)$,

(P_1) $\quad \displaystyle\int_r^1 w(t)\,dt \leqslant C_1(1-r)w(r)$ $\quad$ and

(P_2) $\quad w(r) \leqslant C_2 w(sr + 1 - s)$.

If w is normal we extend it to B by $w(z) = w(|z|)$. For $0 < p < \infty$ the weighted Bergman space $L_{a,w}^p(B)$ is the space of all functions $f \in H(B)$ such that

$$\|f\|_{p,w}^p = \int_B \left|f(z)\right|^p w(z)\,dm(z) < +\infty.$$

In what follows C, C_1, C_2 will stand for positive constants whose value may change from line to line. The expression $A \simeq B$ means $C_1 A \leqslant B \leqslant C_2 A$.

This paper is organized as follows. In Section 2 we will develop some properties of the normal weight, which will be used in the proof of the main results. There are two kinds of normal weights relative to the study of weighted Bergman space theory. One is as in [14,18] and the other is defined with conditions (P_1) and (P_2) as in [3] and here. It is obvious that the weights in [14,18] are normal in the present sense. The results in Section 2 will also enable us to see these two kinds of weights are actually the same in the sense they induce the same pth Bergman space (and equivalent norms). In Section 3, we consider the equivalent characterizations, the point evaluation functionals, and the interpolation sequences on $L^p_{a,w}(B)$. We generalize some results from [3,4,8,11,14,18,19]. The last section, Section 4, contains the main theorems of the paper, where we will find a sufficient and necessary condition on $g \in H(B)$ for which T_g is bounded (respectively compact) from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$ for all possible $0 < p, q < \infty$. Our results will generalize [3,14,18].

Before ending this section we give three examples of the normal weight functions to illustrate how complex these functions can be.

Example 1. $w(r) = (1 - r^2)^\alpha$, $\alpha > -1$.

Example 2. $w(r) = (\log\log(e/(1 - r)))^\alpha$, where $\alpha > 0$ (see [3]).

Example 3. Take $w(r)$ to be Lebesgue integrable on $[0, 1/2)$ and $w(r) \geqslant 1$. For $m \geqslant 1$, define $w(r)$ on $[1 - 1/2^m, 1 - 1/2^{m+1})$ by $w(r) = w(2(r - 1/2))$ inductively.

2. Some results about normal weights

Let $w(r)$ be a normal weight, for $r \in [0, 1)$, we write $\rho(r) = sr + 1 - s$ and

$$w^*(r) = \frac{1}{1-r} \int_r^{\rho(r)} w(t)\,dt.$$

Given two weight functions w_1 and w_2, we say they are equivalent if for $p > 0$ there exist constants C_1 and C_2 such that

$$C_1 \|f\|_{p,w_1} \leqslant \|f\|_{p,w_2} \leqslant C_2 \|f\|_{p,w_1}$$

for all $f \in H(B)$.

Proposition 1. *Given a normal weight w we have the following*:

(i) *If* $0 \leqslant r_1 < r_2 \leqslant \rho(r_1)$, *then*

$$C_1 \leqslant \frac{w^*(r_1)}{w^*(r_2)} \leqslant C_2. \tag{3}$$

(ii) $$w^*(r) \simeq \frac{\int_r^1 w^*(t)\,dt}{1-r}. \tag{4}$$

(iii) $w^*(r)$ *is normal.*
(iv) $w^*(r)$ *is equivalent to* $w(z)$ *if* $|z| = r$.

Proof. (i) It is trivial that

$$1 - r_2 \leqslant 1 - r_1 \leqslant C(1 - r_2).$$

By changing variable in the integral on $[\rho(r_1), \rho(r_2)]$ and applying (P_2), we obtain

$$\begin{aligned} w^*(r_2) &= \frac{1}{1-r_2} \int_{r_2}^{\rho(r_2)} w(t)\,dt = \frac{1}{1-r_2} \left[\int_{r_2}^{\rho(r_1)} + \int_{\rho(r_1)}^{\rho(r_2)} \right] w(t)\,dt \\ &\geqslant \frac{C}{1-r_2} \left[\int_{r_2}^{\rho(r_1)} + \int_{r_1}^{r_2} \right] w(t)\,dt \geqslant C w^*(r_1). \end{aligned}$$

On the other hand, by the estimate above and (P_1),

$$\begin{aligned} w^*(r_2) &\leqslant C w^*\bigl(\rho(r_1)\bigr) = C \frac{\int_{\rho(r_1)}^{\rho(\rho(r_1))} w(t)\,dt}{1-\rho(r_1)} \leqslant C \inf_{u \in [r_1, \rho(r_1)]} \frac{\int_u^1 w(t)\,dt}{1-u} \\ &\leqslant C \inf_{u \in [r_1, \rho(r_1)]} w(u) \leqslant C \frac{\int_{r_1}^{\rho(r_1)} w(t)\,dt}{\rho(r_1) - r_1} \leqslant C \frac{\int_{r_1}^{\rho(r_1)} w(t)\,dt}{1 - r_1}. \end{aligned}$$

This gives $w^*(r_2) \leqslant C w^*(r_1)$.

(ii) By (P_1) again,

$$w^*(r) = \frac{1}{1-r} \int_r^{\rho(r)} w(t)\,dt \leqslant \frac{1}{1-r} \int_r^1 w(t)\,dt \leqslant C w(r). \tag{5}$$

This yields

$$\begin{aligned} \int_r^1 w^*(t)\,dt &\leqslant C \int_r^1 w(t)\,dt \leqslant \int_r^{\rho(r)} w(t)\,dt + C \inf_{u \in [r, \rho(r)]} w(u)(1-u) \\ &\leqslant w^*(r)(1-r) + C \int_r^{\rho(r)} w(t)\,dt = C w^*(r)(1-r). \end{aligned}$$

It gives one direction of (4). The other direction comes from (3),

$$w^*(r) \leqslant C \frac{1}{1-r} \int_r^{\rho(r)} w^*(t)\,dt \leqslant C \frac{1}{1-r} \int_r^1 w^*(t)\,dt.$$

(iii) It is a direct consequence of (3) and (4).

(iv) For a non-negative integer k, denote $\rho_0 = 0$, $\rho_k = \rho(\rho_{k-1})$. Then $\lim_{k\to\infty} \rho_k = 1$. For $f \in H(B)$, by the monotonicity of the mean $M_p(f, r) = \{\int_{\partial B} |f(r\zeta)|^p\,d\sigma(\zeta)\}^{1/p}$ and (i),

$$\begin{aligned}
\|f\|_{p,w}^p &= \int_0^1 M_p^p(f,r)w(r)r^{2n-1}\,dr = \sum_{k=0}^{\infty} \int_{\rho_k}^{\rho_{k+1}} M_p^p(f,r)w(r)r^{2n-1}\,dr \\
&\leqslant \sum_{k=0}^{\infty} M_p^p(f,\rho_{k+1}){\rho_{k+1}}^{2n+1} \int_{\rho_k}^{\rho_{k+1}} w(r)\,dr \\
&= C\sum_{n=0}^{\infty} M_p^p(f,\rho_{k+1}){\rho_{k+1}}^{2n+1} w^*(\rho_k)(1-\rho_k) \\
&\leqslant C\sum_{n=0}^{\infty} M_p^p(f,\rho_{k+1}){\rho_{k+1}}^{2n+1} \int_{\rho_{k+1}}^{\rho_{k+2}} w^*(r)\,dr \\
&\leqslant C\sum_{k=1}^{\infty} \int_{\rho_{k+1}}^{\rho_{k+2}} M_p^p(f,r)w^*(r)r^{2n-1}\,dr \leqslant C\|f\|_{p,w^*}^p.
\end{aligned}$$

The other inequality $\|f\|_{p,w^*}^p \leqslant C\|f\|_{p,w}^p$ comes from (5). The proof is completed. □

Remark 1. The function $w^*(r)$ is continuous. We can construct infinitely many differentiable functions $\phi(r)$ such that $C_1 \leqslant \phi(r)/w*(r) \leqslant C_2$. $\phi(r)$ is obviously normal and equivalent to $w(r)$. To produce $\phi(r)$ we let $\psi(r)$ be the step function on $[0,1)$ such that

$$\psi(r) = \rho_k \quad \text{if } r \in [\rho_k, \rho_{k+1}),\ k = 0, 1, 2, \ldots.$$

From (3) we know $C_1 \leqslant \psi(r)/w*(r) \leqslant C_2$. Now $\phi(r)$ can be easily constructed by adjusting $\psi(r)$ near each point ρ_k.

Proposition 2. *There exists normal weight $\varphi(r)$ such that*

(i) $C_1 \leqslant \dfrac{\varphi(r)}{w^*(r)} \leqslant C_2$.
(ii) *For some constants $-1 < a < b$,*

$$\frac{\varphi(r)}{(1-r)^a} \downarrow 0, \qquad \frac{\varphi(r)}{(1-r)^b} \uparrow \infty \quad \text{as } r \to 1^-. \tag{6}$$

Proof. Take $\varphi(r)$ to be continuous differentiable on $[0,1)$,

$$\varphi(r) = \frac{1}{1-r}\int_r^1 w^*(t)\,dt.$$

Then (i) comes from (4). Furthermore,

$$\left\{\frac{\varphi(r)}{(1-r)^k}\right\}' = \frac{-w^*(r)(1-r) + (1+k)\int_r^1 w^*(t)\,dt}{(1-r)^{2+k}}. \tag{7}$$

From (4) again we have some $k > -1$ such that the right-hand side of (7) is negative. Then $a = (k-1)/2$ will give $\varphi(r)/(1-r)^a \downarrow 0$. Similarly, choose some $k > -1$ such that the right-hand side of (7) is positive. Then any $b > k$ will give $\varphi(r)/(1-r)^b \uparrow \infty$. $\square$

Remark 2. Propositions 1 and 2 show that the two kinds of normality mentioned in Section 1 are the same, in the sense that they induce the same pth Bergman space with equivalent norms.

3. Some properties of the space $L^p_{a,w}(B)$

In this section we exhibit some properties which will not only be used in the proof of the main results but also have their own interest.

3.1. The equivalent norms on $L^p_{a,w}(B)$

For $f \in H(B)$, the radial derivative of f is

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}.$$

For $m = 1, 2, \ldots,$ write $\Re^m f(z) = \Re(\Re^{m-1} f(z))$, $|\operatorname{grad}_m f(z)| = \sum_{|\alpha|=m} |\partial^\alpha f/\partial z^\alpha|$.

Lemma 1. *Let $1 \leqslant p < \infty$, $\delta > 0$, $h : [0,1) \to [0,+\infty)$ measurable, then*

$$\int_0^1 w^*(r)\left\{\int_0^r (r-t)^{\delta-1} h(t)\,dt\right\}^p dr \leqslant C\int_0^1 (1-r)^{p\delta-1} h^p(r) w^*(r)\,dr.$$

Proof. By Proposition 2, we only need to prove

$$\int_0^1 \varphi(r)\left\{\int_0^r (r-t)^{\delta-1} h(t)\,dt\right\}^p dr \leqslant C\int_0^1 (1-r)^{p\delta-1} h^p(r)\varphi(r)\,dr. \tag{8}$$

To prove (8), we take a in Proposition 2 and apply the theorem of [6, p. 758],

$$\begin{aligned}
&\int_0^1 \varphi(r)\left\{\int_0^r (r-t)^{\delta-1} h(t)\,dt\right\}^p dr \\
&\quad\leqslant C\int_0^1 (1-r)^a\left\{\int_0^r (r-t)^{\delta-1} h(t)\left[\frac{\varphi(t)}{(1-t)^a}\right]^{1/p} dt\right\}^p dr \\
&\quad\leqslant C\int_0^1 (1-r)^{a+p\delta-1}\left\{h(r)\left[\frac{\varphi(r)}{(1-r)^a}\right]^{1/p}\right\}^p dr
\end{aligned}$$

$$= C \int_0^1 (1-r)^{p\delta-1} h^p(r)\varphi(r)\,dr.$$

The lemma is proved. □

Theorem 1. *Let $p > 0$ and let m be a positive integer. Then for any function $f \in H(B)$,*

$$\|f\|_{p,w}^p \simeq \sum_{j=0}^{m-1} \left|\mathrm{grad}_j\, f(0)\right|^p + \int_B \left|\Re^m f(z)\right|^p \left(1-|z|^2\right)^{mp} w(z)\,dm(z). \tag{9}$$

Proof. First we prove

$$\|f\|_{p,w}^p \simeq \left|f(0)\right|^p + \int_B \left|\Re f(z)\right|^p \left(1-|z|^2\right)^p w(z)\,dm(z). \tag{10}$$

For $p > 0$ fixed, it is obvious that

$$\left|f(0)\right|^p \leqslant C\|f\|_{p,w}^p, \quad f \in H(B).$$

From [5, p. 80] and [12, p. 14] we have

$$\left(1-r^2\right)^p M_p^p(\Re f, r) \leqslant C M_p^p\bigl(f, \rho(r)\bigr).$$

Notice $(1-r^2)^p w(r)$ is also a normal weight, and

$$\left[\left(1-r^2\right)^p w(r)\right]^* \simeq \left(1-r^2\right)^p w^*(r). \tag{11}$$

Applying Proposition 1,

$$\begin{aligned}
&\int_B \left|\Re f(z)\right|^p \left(1-|z|^2\right)^p w(z)\,dm(z) \\
&\quad\leqslant C \int_B \left|\Re f(z)\right|^p \left(1-|z|^2\right)^p w^*(z)\,dm(z) \\
&\quad= C \int_0^1 M_p^p(\Re f, r)\left(1-r^2\right)^p w^*(r) r^{2n-1}\,dr \\
&\quad\leqslant C \int_0^1 M_p^p\bigl(f, \rho(r)\bigr) w^*(r) r^{2n-1}\,dr \\
&\quad\leqslant C \int_0^1 M_p^p\bigl(f, \rho(r)\bigr) w^*\bigl(\rho(r)\bigr) \rho(r)^{2n-1}\,dr \\
&\quad\leqslant C \int_0^1 M_p^p(f, r) w^*(r) r^{2n-1}\,dr \leqslant C\|f\|_{p,w}^p.
\end{aligned}$$

That gives one direction of (9). To get the reverse inequality, we suppose $f(0)=0$ temporarily and prove

$$\|f\|_{p,w}^p \leqslant C \int_B |\Re f|^p(z)\big(1-|z|^2\big)^p w(z)\,dm(z). \tag{12}$$

If $0<p<1$, Proposition 1 and [13, Theorem 1] yields

$$\begin{aligned}
\|f\|_{p,w}^p &\leqslant C \int_0^1 M_p^p(f,r)w^*(r)r^{2n-1}\,dr \\
&\leqslant C \int_0^1 w^*(r)r^{2n-1}\,dr \int_0^r (r-t)^{p-1} M_p^p(\Re f,t)\,\frac{dt}{r^p} \\
&\leqslant C \int_0^1 M_p^p(\Re f,t)\,dt \int_t^1 (r-t)^{p-1} w^*(r)\,dr \\
&\leqslant C \int_0^1 M_p^p(\Re f,t)\left\{ \int_t^{\rho(t)} + \int_{\rho(t)}^1 \right\}(r-t)^{p-1} w^*(r)\,dr\,dt \\
&\leqslant C \int_0^1 M_p^p(\Re f,t)\left\{ w^*\big(\rho(t)\big) \int_t^{\rho(t)} (r-t)^{p-1}\,dr + \big(\rho(t)-t\big)^{p-1} \int_{\rho(t)}^1 w^*(r)\,dr \right\} dt \\
&\leqslant C \int_0^1 M_p^p(\Re f,t)\big\{ w^*\big(\rho(t)\big)\big(\rho(t)-t\big)^p + \big(\rho(t)-t\big)^{p-1} w^*\big(\rho(t)\big)\big(1-\rho(t)\big)\big\}\,dt \\
&\leqslant C \int_0^1 M_p^p(\Re f,t)(1-t)^p w^*\big(\rho(t)\big)\,dt \leqslant C \int_0^1 M_p^p(\Re f,t)(1-t)^p w^*(t)\,dt \\
&\leqslant C \int_B |\Re f|^p(z)\big(1-|z|^2\big)^p w^*(z)\,dm(z).
\end{aligned}$$

This gives (12). For $1 \leqslant p < \infty$, applying Lemma 1 and [13, Theorem 1] again

$$\begin{aligned}
\|f\|_{p,w}^p &\leqslant C \int_0^1 M_p^p(f,r)w^*(r)r^{2n-1}\,dr \leqslant C \int_0^1 w^*(r)\left\{ \int_0^r M_p(\Re f,t)\,dt \right\}^p dr \\
&\leqslant C \int_0^1 M_p^p(\Re f,r)(1-r)^p w^*(r)\,dr,
\end{aligned}$$

which gives (12) as well. Now for general $f(z)$,

$$\|f\|_{p,w} \leqslant C\big[\big\|f - f(0)\big\|_{p,w} + \big|f(0)\big|\big]$$
$$\leqslant C\left\{\big|f(0)\big| + \left\{\int_B |\Re f|^p(z)\big(1-|z|^2\big)^p w(z)\,dm(z)\right\}^{1/p}\right\}.$$

For $m \geqslant 1$, similar to (11), $(1-r)^{pk}w(r)$ is normal and $[(1-r)^{pk}w(r)]^* \simeq (1-r)^{pk}w^*(r)$. Therefore, by induction, we have (9) from (10). The theorem is proved. □

Remark 3. With the same approach we can prove

$$\|f\|_{p,w}^p \simeq \sum_{j=0}^{m-1}\big|\mathrm{grad}_j\, f(0)\big|^p + \int_B \big|\mathrm{grad}_m\, f(z)\big|^p\big(1-|z|^2\big)^{mp} w(z)\,dm(z).$$

3.2. *The point evaluation functional on $L^p_{a,w}(B)$*

Let $\beta(\cdot,\cdot)$ denote the Bergman metric on B. The Bergman ball $E(z,r)$ with center $z \in B$ and radius $r > 0$ is defined as $E(z,r) = \{\zeta \in B\colon\ \beta(\zeta,z) < r\}$. It is well known that

$$\big|E(z,r)\big| \simeq r^{2n}\big(1-|z|^2\big)^{n+1}, \qquad 1-|\zeta| \simeq 1-|z| \quad \text{for } \zeta \in E(z,r). \tag{13}$$

For $r > 0$ fixed, by Proposition 1 there is some constant $C(r)$ such that for $z \in B$ and $\zeta \in E(z,r)$

$$\frac{1}{C(r)} \leqslant \frac{w^*(\zeta)}{w^*(z)} \leqslant C(r). \tag{14}$$

Theorem 2. *The point evaluation functional Λ_z on $L^p_{a,w}(B)$ defined by*

$$\Lambda_z(f) = f(z), \quad f \in L^p_{a,w}(B)$$

is bounded with the norm estimate

$$\|\Lambda_z\| \simeq \frac{1}{[w^*(z)(1-|z|^2)^{n+1}]^{1/p}}.$$

Proof. Given $z \in B$, by the plurisubharmonicity of $|f(z)|^p$ and (14) we have

$$|f(z)|^p \leqslant C\frac{1}{|E(z,1)|}\int_{E(z,1)} \big|f(\zeta)\big|^p\,dm(\zeta)$$
$$\leqslant C\frac{1}{|E(z,1)|w^*(z)}\int_{E(z,1)} \big|f(\zeta)\big|^p w^*(\zeta)\,dm(\zeta). \tag{15}$$

Then (13) gives

$$\|\Lambda_z\| \leqslant C\frac{1}{[w^*(z)(1-|z|^2)^{n+1}]^{1/p}}. \tag{16}$$

Now for given ζ we consider the function $f_\zeta(z)$ with fixed $\beta > b$,

$$f_\zeta(z) = \left\{ \frac{(1-|\zeta|^2)^\beta}{w^*(\zeta)(1-\langle z,\zeta\rangle)^{n+1+\beta}} \right\}^{1/p}. \tag{17}$$

Then

$$f_\zeta(\zeta) = \frac{1}{[w^*(\zeta)(1-|\zeta|^2)^{n+1}]^{1/p}}. \tag{18}$$

We claim $\|f_\zeta\|_{p,w} \leqslant C$, and this together with (16) and (18) will end the proof of the theorem. In fact, applying [15, Lemma 6],

$$\begin{aligned}
\|f_\zeta\|_{p,w}^p &\leqslant C\frac{(1-|\zeta|^2)^\beta}{w^*(\zeta)} \int_0^1 \varphi(r) \int_{\partial B} \frac{1}{|1-\langle rz,\zeta\rangle|^{n+1+\beta}}\, d\sigma(z)\, dr \\
&\leqslant C\frac{(1-|\zeta|^2)^\beta}{w^*(\zeta)} \int_0^1 \frac{\varphi(r)}{(1-r|\zeta|)^{1+\beta}}\, dr \\
&= C\frac{(1-|\zeta|^2)^\beta}{w^*(\zeta)} \left\{ \int_0^{|\zeta|} + \int_{|\zeta|}^1 \right\} \frac{\varphi(r)}{(1-r|\zeta|)^{1+\beta}}\, dr \\
&\leqslant C\frac{(1-|\zeta|^2)^\beta}{w^*(\zeta)} \left\{ \frac{w^*(\zeta)}{(1-|\zeta|)^b} \int_0^{|\zeta|} \frac{(1-r)^b\, dr}{(1-r|\zeta|)^{1+\beta}} \right. \\
&\qquad \left. + \frac{w^*(\zeta)}{(1-|\zeta|)^a} \int_{|\zeta|}^1 \frac{(1-r)^a\, dr}{(1-r|\zeta|)^{1+\beta}} \right\} \\
&\leqslant C. \qquad \square
\end{aligned} \tag{19}$$

3.3. *The interpolation sequence in $L^p_{a,w}(B)$*

Lemma 2. *Suppose $\beta > b$ is in Proposition 2. Then for any $\varepsilon > 0$ there is some constant K such that for any sequence $\{\zeta_j\} \subseteq B$,*

$$\inf\{\beta(\zeta_j,\zeta_k):\ j,k = 1,2,\ldots,\ j \neq k\} \geqslant K$$

we have

$$\sum_{j=1,\, j\neq k}^\infty \frac{(1-|\zeta_j|^2)^{n+1} w^*(\zeta_j)}{|1-\langle \zeta_j,\zeta_k\rangle|^{n+1+\beta}} < \varepsilon \frac{w^*(\zeta_k)}{(1-|\zeta_k|^2)^\beta} \tag{20}$$

for $k = 1,2,\ldots.$

Proof. First, we prove that if K is large enough, then

$$\sup_{z\in B}\int\limits_{B\setminus E(z,K)}\frac{w^*(\zeta)}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)<\varepsilon\frac{w^*(z)}{(1-|z|^2)^\beta}. \tag{21}$$

In fact, given any $-1<c<\beta$, a direct calculation using [12, Theorems 2.22 and 2.26] gives

$$\sup_{z\in B}\int\limits_{B\setminus E(z,K)}\frac{(1-|\zeta|^2)^c}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)<\varepsilon\frac{(1-|z|^2)^c}{(1-|z|^2)^\beta}$$

if K is large enough. Set $B_{|z|}=\{\zeta\in B\colon\ |\zeta|<|z|\}$. Then, similar to the estimate (19), we know

$$\begin{aligned}
&\int\limits_{B\setminus E(z,K)}\frac{w^*(\zeta)}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\\
&\quad\leqslant C\int\limits_{B\setminus E(z,K)}\frac{\varphi(r)}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\\
&\quad\leqslant C\left\{\int\limits_{(B\setminus E(z,K))\cap B_{|z|}}+\int\limits_{(B\setminus E(z,K))\setminus B_{|z|}}\right\}\frac{\varphi(r)}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\\
&\quad\leqslant C\left\{\frac{w^*(|z|)}{(1-|z|^2)^b}\int\limits_{(B\setminus E(z,K))\cap B_{|z|}}\frac{(1-|\zeta|^2)^b}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\right.\\
&\qquad\left.+\frac{w^*(|z|)}{(1-|z|^2)^a}\int\limits_{(B\setminus E(z,K))\setminus B_{|z|}}\frac{(1-|\zeta|^2)^a}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\right\}\\
&\quad\leqslant C\left\{\frac{w^*(|z|)}{(1-|z|^2)^b}\int\limits_{B\setminus E(z,K)}\frac{(1-|\zeta|^2)^b}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\right.\\
&\qquad\left.+\frac{w^*(|z|)}{(1-|z|^2)^a}\int\limits_{B\setminus E(z,K)}\frac{(1-|\zeta|^2)^a}{|1-\langle\zeta,z\rangle|^{n+1+\beta}}\,dm(\zeta)\right\}\\
&\quad\leqslant\varepsilon\frac{w^*(z)}{(1-|z|^2)^\beta}.
\end{aligned}$$

Now, if K is large enough,

$$\begin{aligned}
\sum_{j=1,\ j\neq k}^{\infty}\frac{(1-|\zeta_j|^2)^{n+1}w^*(\zeta_j)}{|1-\langle\zeta_j,\zeta_k\rangle|^{n+1+\beta}}&\leqslant C\sum_{j=1,\ j\neq k}^{\infty}\int\limits_{E(\zeta_j,1)}\frac{w^*(\zeta)}{|1-\langle\zeta,\zeta_k\rangle|^{n+1+\beta}}\,dm(\zeta)\\
&=C\int\limits_{\bigcup_{j=1,\ j\neq k}^{\infty}E(\zeta_j,1)}\frac{w^*(\zeta)}{|1-\langle\zeta,\zeta_k\rangle|^{n+1+\beta}}\,dm(\zeta)
\end{aligned}$$

$$\leqslant C \int\limits_{B\setminus E(\zeta_k, K-1)} \frac{w^*(\zeta)}{|1-\langle \zeta, \zeta_k\rangle|^{n+1+\beta}}\, dm(\zeta)$$
$$< \varepsilon \frac{w^*(\zeta_k)}{(1-|\zeta_k|^2)^\beta}. \tag{22}$$

The estimate (20) is proved. □

Theorem 3. *Let $p > 0$. There exists some constant $K > 0$ such that if $\{z_j\}$ is a sequence in B satisfying*

$$\inf\bigl\{\beta(z_j, z_k):\ j, k = 1, 2, \ldots, j \neq k\bigr\} > K, \tag{23}$$

then for any sequence $\{w_j\}$ with

$$\sum_{j=1}^{\infty} |w_j|^p \bigl(1-|z_j|^2\bigr)^{n+1} w^*(z_j) \leqslant 1$$

we have some $f \in L^p_{a,w}(B)$ such that $\|f\|_{p,w} \leqslant C$ and $f(z_j) = w_j$.

Proof. The theorem is a generalization of [11, Theorem, p. 231]. And the proof is along the same line as that in [11]. Given a sequence $\{z_j\}$ satisfying (23), define the operator T from $L^p_{a,w}(B)$ to l^p by

$$Tf = \bigl\{f(z_j)\bigl[\bigl(1-|z_j|^2\bigr)^{n+1} w^*(z_j)\bigr]^{1/p}\bigr\}_{j=1}^{\infty}, \quad f \in L^p_{a,w}(B).$$

As in (20), T is bounded. Furthermore, take

$$f_{z_j}(z) = \left\{\frac{(1-|z_j|^2)^\beta}{w^*(z_j)(1-\langle z, z_j\rangle)^{n+1+\beta}}\right\}^{1/p}.$$

If K is large enough, then the operator R_1 from l^p to $L^p_{a,w}(B)$ defined by $R_1(\{\lambda_j\}) = \sum_{j=1}^{\infty} \lambda_j f_{z_j}(z)$ is bounded and the operator TR_1 is invertible. Obviously, this assertion will give the conclusion of the theorem. That R_1 is bounded and TR_1 is invertible can be proved in two cases $0 < p \leqslant 1$ and $1 < p < \infty$ as in [11]. The key point in the present setting is to apply Lemma 2, and in [11] the corresponding lemma is Lemma 3.1 (a little attention should be paid to that the factor B_{ij}^{-1-r} in the left-hand side of (3.3) there should be read as B_{jj}^{-1-r}). The detail is omitted here. □

Applying Theorem 3 we have the following theorem.

Theorem 4. *Let μ be a positive Borel measure on B,*

$$\hat{\mu}(z) = \frac{\mu(E(z,1))}{w^*(z)(1-|z|^2)^{n+1}}.$$

Let $0 < q \leqslant p < \infty$. Then a necessary and sufficient condition for there to exist a constant G such that

$$\left\{ \int\limits_B \left| f(z) \right|^q d\mu(z) \right\}^{1/q} \leqslant G \left\{ \int\limits_B \left| f(z) \right|^p w^*(z)\, dm(z) \right\}^{1/p} \tag{24}$$

for all $f \in L^p_{a,w}(B)$ *is that*

$$\int\limits_B \hat{\mu}(z)^s w^*(z)\, dm(z) < \infty, \tag{25}$$

where $1/s + q/p = 1$. *Furthermore, we have*

$$\left\{ \int\limits_B \hat{\mu}(z)^s w^*(z)\, dm(z) \right\}^{1/s} \leqslant C G^q. \tag{26}$$

Proof. When $w \equiv 1$ the assertion (25) is just the main result of [8]. Taking normal weights into account, the proof of (25) uses the same approach with the modification that the lemma on [8, p. 128] should be replaced by the following lemma, Lemma 3. And the proof of Lemma 3 can also be carried out as that in [8]. The estimate (26) does not appear explicitly in [8]. A careful check of the constant C in (3.4) there will give (26). □

Lemma 3. *There exists some constant* A *depending only on* q *and* K *such that if* $\{z_j\}$ *satisfies* (23) *and* $\delta < K/4$, *then for every* $f \in L^p_{a,w}(B)$ *with* $0 < q \leqslant p$,

$$\sum_j \int\limits_{E(z_j,\delta)} \left| f(z) - f(z_j) \right|^q d\mu(z) \leqslant A\delta^q \|f\|^q_{p,w} \left\{ \left[\sum_j \mu\bigl(E(z_j,\delta)\bigr) \right]^s \left[w^*(z_j) m\left(E\left(z_j, \frac{K}{2} \right) \right) \right]^{1-s} \right\}^{1/s},$$

where s *is as in Theorem* 4.

4. Main theorems

Following [17], a function $f \in H(B)$ is called a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup\left\{ \left| \Re f(z) \right| \left(1 - |z|^2\right) \colon z \in B \right\} < \infty;$$

and f is called a little Bloch function if

$$\lim_{|z| \to 1} \left| \Re f(z) \right| \left(1 - |z|^2\right) = 0.$$

The set of all Bloch and little Bloch functions will be denoted as $\mathcal{B}$ and $\mathcal{B}_0$, respectively.

Theorem 5. *Let* $p, q > 0$. *Then*

(A) *For* $p > q$, T_g *is bounded from* $L^p_{a,w}(B)$ *to* $L^q_{a,w}(B)$ *if and only if* $g \in L^k_{a,w}(B)$, *where* $1/k = 1/q - 1/p$.

(B) *T_g is bounded on $L^p_{a,w}(B)$ if and only if $g \in \mathcal{B}$.*
(C) *For $p < q$, T_g is bounded from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$ if and only if*

$$\left|\Re g(z)\right| \leqslant C\left[\frac{w^*(z)}{(1-|z|^2)^{k-(n+1)}}\right]^{1/k}, \tag{27}$$

where $1/k = 1/p - 1/q$.

Furthermore, if T_g is bounded from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$, then

$$\|T_g\| \simeq \begin{cases} \|g-g(0)\|_{k,w}, & \text{for } p > q, \\ \|g\|_{\mathcal{B}}, & \text{for } p = q, \\ \sup_{z\in B} |\Re g(z)|\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{1/k}, & \text{for } p < q. \end{cases} \tag{28}$$

Proof. First we prove the sufficiency.

(i) Let $0 < q < p < \infty$, $1/k = 1/q - 1/p$, $f \in L^p_{a,w}(B)$, and $g \in L^k_{a,w}(B)$. Then by Theorem 1 and Hölder's inequality,

$$\begin{aligned}
\|T_g(f)\|_q^q &\leqslant C\int_B \left|\Re T_g(f)(z)\right|^q \left(1-|z|^2\right)^q w^*(z)\,dm(z) \\
&\leqslant C\int_B \left|f(z)\Re g(z)\right|^q \left(1-|z|^2\right)^q w^*(z)\,dm(z) \\
&\leqslant C\left\{\int_B \left[\left|\Re g(z)\right|^q \left(1-|z|^2\right)^q\right]^{k/q} w^*(z)\,dm(z)\right\}^{q/k} \\
&\quad\times\left\{\int_B \left[\left|f(z)\right|^q\right]^{p/q} w^*(z)\,dm(z)\right\}^{q/p} \\
&\leqslant C\left\|g-g(0)\right\|_k^q \|f\|_p^q.
\end{aligned}$$

(ii) Let $0 < p < \infty$, $f \in L^p_{a,w}(B)$, and g be a Bloch function. Then

$$\begin{aligned}
\left\|T_g(f)\right\|_p^p &\leqslant C\int_B \left|\Re T_g(f)(z)\right|^p \left(1-|z|^2\right)^p w^*(z)\,dm(z) \\
&\leqslant C\sup_{z\in B}\left[\left(1-|z|^2\right)\left|\Re g(z)\right|\right]^p \int_B \left|f(z)\right|^p w^*(z)\,dm(z) \\
&\leqslant C\|g\|_{\mathcal{B}}^p \|f\|_p^p.
\end{aligned}$$

(iii) Let $0 < p < q < \infty$, $1/k = 1/p - 1/q$, and $f \in L^p_{a,w}(B)$,

$$\sup_{z\in B}\left|\Re g(z)\right|\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{1/k} < \infty.$$

By Theorem 2,

$$\begin{aligned}
\|T_g(f)\|_q^q &\leqslant C\int_B \left|f(z)\Re g(z)\right|^q\left(1-|z|^2\right)^q w^*(z)\,dm(z)\\
&\leqslant C\|f\|_{p,w}^{q-p}\int_B \left|f(z)\right|^p\left|\Re g(z)\right|^q\left(1-|z|^2\right)^{q-((n+1)(q-p))/p}\\
&\quad\times w^*(r)^{1-(q-p)/p}\,dm(z)\\
&\leqslant C\|f\|_{p,w}^{q-p}\sup_{z\in B}\left\{\left|\Re g(z)\right|\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{1/k}\right\}^q\\
&\quad\times\int_B \left|f(z)\right|^p w^*(z)\,dm(z)\\
&\leqslant C\left\{\sup_{z\in B}\left|\Re g(z)\right|\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{1/k}\right\}^q\|f\|_{p,w}^q.
\end{aligned}$$

The sufficiency and one direction of (28) are proved. To prove the necessity we suppose $0<p,q<\infty$, and T_g is bounded from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$.

(i) If $p<q$, we take f_ζ as in (17) for each $\zeta\in B$. Proposition 1 and the plurisubharmonicity imply

$$\begin{aligned}
\|T_g f_\zeta\|_{q,w}^q &\geqslant C\int_{E(\zeta,1)}\left|f_\zeta(u)\right|^q\left|\Re g(u)\right|^q\left(1-|u|^2\right)^q w^*(u)\,dm(u)\\
&\geqslant C\left|f_\zeta(\zeta)\right|^q\left|\Re g(\zeta)\right|^q\left(1-|\zeta|^2\right)^{q+(n+1)}w^*(\zeta)\\
&= C\left\{\left|\Re g(\zeta)\right|\left[\frac{(1-|\zeta|^2)^{k-(n+1)}}{w^*(\zeta)}\right]^{1/k}\right\}^q.
\end{aligned}$$

This gives

$$\sup_{z\in B}\left|\Re g(z)\right|\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{1/k}\leqslant C\|T_g\|.$$

(ii) If $p=q$, as above we have

$$\begin{aligned}
\|T_g f_\zeta\|_{p,w}^p &\geqslant C\left|f_\zeta(\zeta)\right|^p\left|\Re g(\zeta)\right|^p\left(1-|\zeta|^2\right)^{p+(n+1)}w^*(\zeta)\\
&= C\left[\left|\Re g(\zeta)\right|\left(1-|\zeta|^2\right)\right]^p.
\end{aligned}$$

This implies $\|g\|_{\mathcal{B}}\leqslant C\|T_g\|$.

(iii) If $p>q$, by Theorem 1 we know

$$\left\{\int_B \left|f(u)\right|^q\left|\Re g(u)\right|^q\left(1-|u|^2\right)^q w^*(u)\,dm(u)\right\}^{1/q}\leqslant C\|f\|_{p,w}$$

for all $f\in L^p_{a,w}(B)$. Theorem 4 tells us there is some constant C such that

$$\left\{\int_B \left[\left|\Re g(u)\right|\left(1-|u|^2\right)\right]^{(pq)/(p-q)}w^*(u)\,dm(u)\right\}^{1/q-1/p}\leqslant C\|T_g\|.$$

Thus $g \in L^k_{a,w}(B)$ and

$$\|g - g(0)\|_{k,w} \leqslant C\|T_g\|.$$

The proof of Theorem 5 is completed. □

For $k > 0$, let

$$X_{k,0} = \left\{ f \in H(B)\colon \lim_{|z|\to 1} \left|\Re g(z)\right| \left[\frac{(1-|z|)^{k-(n+1)}}{w^*(z)} \right]^{1/k} = 0 \right\}.$$

Define the metric on $X_{k,0}/C$ by

$$d(f,h) = \sup_{z\in B} \left|\Re f(z) - \Re h(z)\right| \left[\frac{(1-|z|)^{k-(n+1)}}{w^*(z)} \right]^{1/k}.$$

Then $X_{k,0}/C$ is a Banach space.

Lemma 4. *If $X_{k,0}$ is non-trivial, that is $X_{k,0}$ contains some non-constant function, then it contains all polynomials. Furthermore, polynomials are dense in $X_{k,0}$.*

Proof. Suppose $f \in X_{k,0}$ is non-constant. Then $M_\infty(\Re f, r) = \max_{|z|=r} |\Re f(z)|$ is strictly increasing. This gives $M_\infty(\Re f, r) > C > 0$ for $r \in [1/2, 1)$ and

$$\lim_{|z|\to 1} \frac{(1-|z|)^{k-(n+1)}}{w^*(z)} = 0. \tag{29}$$

Now for any polynomial $h(z)$, $\Re h(z)$ is bounded on B. By (29), we know $h \in X_{k,0}$.

Given $f \in X_{k,0}$ and $\varepsilon > 0$, by definition we have some $\eta \in (0,1)$ such that

$$\sup_{\eta \leqslant r < 1} M_\infty(\Re f, r) \left[\frac{(1-r)^{k-(n+1)}}{w^*(r)} \right]^{1/k} < \varepsilon. \tag{30}$$

Then for all $t \in (0,1)$, the function $f_t(z) = f(tz)$ satisfies

$$\begin{aligned} &\sup_{\eta \leqslant r < 1} M_\infty(\Re f_t, r) \left[\frac{(1-r^2)^{k-(n+1)}}{w^*(r)} \right]^{1/k} \\ &\quad = \sup_{\eta \leqslant r < 1} M_\infty(\Re f, tr) \left[\frac{(1-r^2)^{k-(n+1)}}{w^*(r)} \right]^{1/k} \\ &\quad \leqslant \sup_{\eta \leqslant r < 1} M_\infty(\Re f, r) \left[\frac{(1-r^2)^{k-(n+1)}}{w^*(r)} \right]^{1/k} < \varepsilon. \end{aligned} \tag{31}$$

We can fix t sufficiently near 1 such that

$$\max_{|z|\leqslant \eta} \left|\Re f(z) - \Re f_t(z)\right| \left[\frac{(1-|z|)^{k-(n+1)}}{w^*(z)} \right]^{1/k} < \varepsilon. \tag{32}$$

Now by (30)–(32) we have $d(f, f_t) < 3\varepsilon$. That there exists some polynomial $h(z)$ such that $d(f_t, h) < \varepsilon$ is trivial. Therefore, polynomials are dense in $X_{k,0}$. □

Theorem 6. *Let $p, q > 0$.*

(A) *For $p > q$, T_g is compact from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$ if and only if $g \in L^k_{a,w}(B)$, where $1/k = 1/q - 1/p$.*
(B) *T_g is compact on $L^p_{a,w}(B)$ if and only if $g \in \mathcal{B}_0$.*
(C) *For $p < q$, T_g is compact from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$ if and only if $g \in X_{k,0}$, where $1/k = 1/p - 1/q$.*

Proof. We prove the sufficiency first. Suppose g is non-constant, otherwise there is nothing to prove.

(i) If $0 < p < q < \infty$, then Lemma 4 tells us that g can be approximated by polynomials with the metric $d(\cdot,\cdot)$. By Theorem 5, to prove T_g is compact we need only prove the operator T_P with polynomial symbol P is compact. Notice that T_P is the product of the bounded operator $M_{\Re P}$ from $L^p_{a,w}(B)$ to $L^p_{a,w}(B)$ defined by

$$M_{\Re P}(f)(z) = f(z)\Re P(z)$$

and the operator T defined by

$$T(f)(z) = \int_0^1 \frac{f(tz) - f(0)}{t}\,dt, \qquad f \in H(B).$$

Hence it suffices to prove T is compact from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$.

Now, given any sequence $\{f_m\}$ in $L^p_{a,w}(B)$ satisfying

$$\|f_m\|_{p,w} \leqslant 1, \qquad f_m(z) \to 0 \tag{33}$$

uniformly on compact subsets of B, we are going to prove

$$\lim_{m\to\infty} \big\|T(f_m)\big\|_{q,w} = 0. \tag{34}$$

For any $\varepsilon > 0$ by (29) we can choose $\eta \in (0,1)$ such that for $\eta \leqslant r < 1$,

$$\frac{(1-r)^{k-(n+1)}}{w^*(r)} < \varepsilon.$$

Therefore,

$$\begin{aligned}
&\big\|T(f_m)\big\|^q_{q,w}\\
&\quad \leqslant C \int_B \big|\Re\big(T(f_m)(z)\big)\big|^q \big(1-|z|^2\big)^q w^*(z)\,dm(z)\\
&\quad = C \int_B \big|f_m(z) - f_m(0)\big|^q \big(1-|z|^2\big)^q w^*(z)\,dm(z)\\
&\quad \leqslant C\bigg\{\int_B \big|f_m(z)\big|^q \big(1-|z|^2\big)^q w^*(z)\,dm(z) + \big|f(0)\big|^q\bigg\}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C\left\{\|f_m\|^{q-p}\int_B\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{q/k}\left|f_m(z)\right|^p w^*(z)\,dm(z)+\left|f(0)\right|^q\right\}\\
&\leqslant C\left\{\|f_m\|^{q-p}\left[\int_{B_\eta}+\int_{B\setminus B_\eta}\right]\left[\frac{(1-|z|^2)^{k-(n+1)}}{w^*(z)}\right]^{q/k}\left|f_m(z)\right|^p w^*(z)\,dm(z)\right.\\
&\qquad\left.+\left|f(0)\right|^q\right\}\\
&\leqslant C\left\{\|f_m\|^{q-p}\left[\sup_{z\in B_\eta}\left|f_m(z)\right|^p+\varepsilon^{q/k}\|f_m\|^p_{p,w}\right]+\left|f(0)\right|^q\right\}\\
&\leqslant C\left\{\sup_{z\in B_\eta}\left|f_m(z)\right|^p+\varepsilon^{q/k}+\left|f(0)\right|^q\right\}.
\end{aligned}$$

From this and (33) we obtain (34). This implies T is compact from $L^p_{a,w}(B)$ to $L^q_{a,w}(B)$.

(ii) If $p=q$, we also need prove (34) because any little Bloch function g can be approximated by polynomials in the Bloch norm $\|\cdot\|_{\mathcal{B}}$ (see [17]). Then, as above,

$$\begin{aligned}
\left\|T(f_m)\right\|^p_{q,w}&\leqslant C\int_B\left|f_m(z)-f_m(0)\right|^p\left(1-|z|^2\right)^p w^*(z)\,dm(z)\\
&\leqslant C\left\{\left[\int_{B_\eta}+\int_{B\setminus B_\eta}\right]\left|f_m(z)\right|^p\left(1-|z|^2\right)^p w^*(z)\,dm(z)+\left|f(0)\right|^p\right\}.
\end{aligned}$$

From this (34) follows.

(iii) If $0<q<p<\infty$, by Hölder inequality,

$$\begin{aligned}
&\left\|T(f_m)\right\|^q_{q,w}\\
&\quad\leqslant C\left\{\int_B\left|f_m(z)\right|^q\left(1-|z|^2\right)^q w^*(z)\,dm(z)+\left|f_m(0)\right|^q\right\}\\
&\quad\leqslant C\left\{\left[\int_B\left|f_m(z)\right|^p\left(1-|z|^2\right)^p w^*(z)\,dm(z)\right]^{q/p}+\left|f_m(0)\right|^q\right\}\\
&\quad\leqslant C\left\{\sup_{z\in B_t}\left|f_m(z)\right|^q+\left[\int_{B\setminus B_t}\left|f_m(z)\right|^p\left(1-|z|^2\right)^p w^*(z)\,dm(z)\right]^{q/p}+\left|f_m(0)\right|^q\right\}\\
&\quad\leqslant C\left\{\sup_{z\in B_t}\left|f_m(z)\right|^q+\left(1-t^2\right)^q+\left|f_m(0)\right|^q\right\},
\end{aligned}$$

here C is independent of $t\in[0,1)$. From this we obtain (34).

To prove the necessity, we need only consider the case $0<p\leqslant q<\infty$. For $\zeta\in B$ take $f_\zeta(z)$ as (17). As $0<p<q<\infty$, the compactness of T_g gives

$$\left\{\left|\Re g(\zeta)\right|\left[\frac{(1-|\zeta|^2)^{k-(n+1)}}{w^*(\zeta)}\right]^{1/k}\right\}^q$$

$$
\begin{aligned}
&= \left|f_\zeta(\zeta)\right|^q \left|\Re g(\zeta)\right|^q w^*(\zeta)\left(1-|\zeta|^2\right)^{n+1+q} \\
&\leqslant C \int\limits_{E(\zeta,1)} \left[\left|f_\zeta(z)\Re g(z)\right|\right]^q \left(1-|z|^2\right)^q w^*(z)\,dm(z) \\
&\leqslant C \int\limits_{B} \left[\left|f_\zeta(z)\Re g(z)\right|\right]^q \left(1-|z|^2\right)^q w^*(z)\,dm(z) \\
&\leqslant C T_g(f_\zeta) \to 0
\end{aligned}
$$

as $|\zeta| \to 1$. For the case $p = q$, the proof is similar, and so is omitted here. The proof is completed. □

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